Wrong-Way maps in KK-theory.

Definition. Let X and Y be G-spaces, G cpct.

A <u>normally nonsingular map</u> from X to Y consist of:

1. a subtrivial G-bundle N over X
2. a finite dimensional representation $\pi : G \to GL(V)$
3. an open embedding $f : N \to Y \times V$.

Such a map is said to be <u>K-oriented</u> if $N \to X$ and $Y \times V \to Y$ are K-oriented.

Then iso for K-theory.

$$\begin{tikzcd} N \arrow[r, hook, "\text{open}"] \arrow[d, "K\text{-oriented}"'] & Y \times V \arrow[d, "K\text{-oriented}"] \\ X \arrow[r, dashed, "f"', "\text{trace}" below] & Y \end{tikzcd}$$

Such a map induces a wrong-way map

$$f_! : K_G^*(X) \longrightarrow K_G^{*+d}(Y) \quad , \quad d = \dim N - \dim V,$$

by composing the Thom isomorphism for $N \to X$ and $Y \times V \to Y$ and the wrong-way map for the open embedding $N \hookrightarrow Y \times V$.

Example.

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d] & \mathbb{C}^n \arrow[d] \\ TX \arrow[r, dashed, "f"] & pt \end{tikzcd}$$

leads to the G-equivariant topological index map

$$f_! = \mathrm{ind}_G : K^0_G(TX) \longrightarrow K^0_G(pt) = R(G).$$

How to lift smooth maps to normally nonsingular maps?

X, Y smooth G-manifolds, $N \to X$ smooth, $N \hookrightarrow Y \times V$ diffeo onto its range.

Theorem. [Mostow] A G-equivariant embedding of X into a finite-dimensional representation exists iff the G-action on X has finite orbit type

(i.e. only finitely many conjugacy classes of subgroups of G occur as stabilizers of points in X)

Remark. X cpct $\Rightarrow$ X has finite orbit type.

Exercise 44. Show that the $\mathbb{T}$-space

$$X = \coprod_{n=1}^{\infty} \mathbb{T} / \{ e^{2\pi i k/n} \mid k = 0, \dots, n-1 \}$$

does not embed $\mathbb{T}$-equivariantly into any $\mathbb{T}$-repr.

Solution. By Mostow theorem, the set of conjugacy classes of stabilizers must be finite. But $\mathbb{T}$ abelian, and stabilizers are $\{ e^{2\pi i k/n} \mid k = 0, \dots, n-1 \} \cong \mathbb{Z}/n\mathbb{Z}$ all different for $n = 1, 2, \dots$. ↯ $\square$

Warning. For noncompact groups or groupoids the Mostow embedding may fail.

A normally nonsingular map from Mostow embedding.

$h : X \hookrightarrow V$ a smooth G-equivariant embedding into a representation of G.

$\Rightarrow$ $(f, h) : X \hookrightarrow Y \times V$ also an embedding

$\Rightarrow$ the tubular neighborhood $N \hookrightarrow Y \times V$ an open embedding.

This leads to a normally nonsingular map f with a trace.

Up to the equivalence up to the modification by lifting by a representation W of G

$$
\begin{tikzcd}
N \oplus (X \times W) \arrow[r, hook] \arrow[d] & Y \times (V \oplus W) \arrow[d] \\
X \arrow[r, dashed, "f^W"] & Y
\end{tikzcd}
$$

$f^W \sim f$, and on isotopy:

$$
\begin{tikzcd}
N \times [0,1] \arrow[r, hook] \arrow[d] & (Y \times [0,1]) \arrow[d] \\
X \times [0,1] \arrow[r, dashed, "F"] \arrow[dr] & Y \times [0,1] \arrow[d] \\
& {[0,1]}
\end{tikzcd}
$$

$F_0 \sim F_1$

Theorem. If X is of finite G-orbit type, then equivalence classes of normally nonsingular maps correspond to homotopy classes of smooth maps.

Composition of Normally Nonsingular Maps.

$$
\begin{tikzcd}
N \arrow[r, hook] \arrow[d] & Y \times V \arrow[d] \\
X \arrow[r, dashed, "f"] & Y
\end{tikzcd}
\quad
\begin{tikzcd}
M \arrow[r, hook] \arrow[d] & Z \times W \arrow[d] \\
Y \arrow[r, dashed, "g"] & Z
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
N \times f^* M \arrow[r, hook] \arrow[d] & V \times M \arrow[r, hook] \arrow[d] & Z \times V \times W \arrow[d] \\
X & Y & Z
\end{tikzcd}
$$

induces $\quad K_G^*(X) \to K_G^*(Y) \longrightarrow K_G^*(Z)$, because lifting does not alter the induced map on K-theory and neither going up by the pull-back and going down along a K-oriented vector bundle by the Thom isomorphism.

Remark. In fact, wrong-way map construction yields classes in $KK^G_*(C_0(X), C_0(Y))$ and the composition of normally nonsingular maps goes to the Kasparov product.

theorem. [the Atiyah–Singer Index thm for families]
Let $\pi : X \to Y$ be a G-equivariant submersion.
Then $\pi_! \in KK^G_d(C_0(X), C_0(Y))$, where $d = \dim X - \dim Y$,
is equal to the class of the family of Dirac operators along the fibers of π.

Proof. [Sketch of] Lift π to a normally nonsingular map

$$\begin{tikzcd} N \arrow[r, hook, "f"] \arrow[d, "\sigma"'] & Y \times V \arrow[d, "\tau"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

For vector bundle projections, the construction of the Thom isomorphism in KK^G gives $\sigma_! = [D_\sigma]$, and this class is invertible. The map $\tau \circ f = \pi \circ \sigma$ is a vector bundle projection followed by an open embedding. The Dirac class is compatible with open embedding. Hence

$$(\pi \circ \sigma)_! = [D_{\pi \circ \sigma}] = [D_\pi] \circ [D_\sigma]$$

Finally, the explicit computation of the Kasparov product gives the Atiyah-Singer formula. $\square$

Fredholm modules.

Definition. $\mathbf{Fred}$:= space of Fredholm operators on a separable Hilbert space.

1. **Atiyah-Jänich theorem.** For any compact Hausdorff space X, there is a natural isomorphism of groups

$$\mathrm{ind} : [X, \mathbf{Fred}] \longrightarrow K^0(X)$$

such that for $X = \{pt\}$ is an index of a Fredholm operator.

2. Noncommutative Atiyah-Jänich theorem.

Theorem. [Kuiper-Mingo] If A is a σ-unital C^*-algebra and $\mathcal{H}$ is a separable right A-Hilbert module, the unitary group of $\mathrm{End}_A(\mathcal{H})$ is contractible.

Definition. $\mathbf{Fred}_A :=$ space of generalized Fredholm operators on a separable right A-module $\mathcal{H}$ (i.e. invertible modulo the ideal of generalized compact operators on $\mathcal{H}$).

Theorem. [Mingo] If A is a σ-unital C^*-algebra, the index map induces a natural isomorphism of groups

$$\mathrm{ind} : [X, \mathbf{Fred}_A] \longrightarrow K_0(C(X,A)).$$

__Recall:__ Cycles of K-homology are triples (H, F, φ) $F \in \mathbf{Fred}$, $\varphi : A \to B(H)$, $[F, \varphi(A)] \subset K(H)$, $F^2 = 1 \bmod K(H)$, $F^* = F \bmod K(H)$. (Graded for even, non-graded for odd K-homology.)